

# Multi-Channel Bethe-Salpeter Equation\*

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## Abstract

A general form of multichannel Bethe-Salpeter equation is considered. A set of relations which may be helpful in approximate treatments is given. An example of extracting useful information from the equations is discussed: we consider the most general trilinear coupling of  $N$  different scalar fields and obtain - in the ladder approximation - closed expressions for the Regge trajectories and their couplings to different channels in the vicinity of  $\ell = -1$ . Sum rules and an example containing non-obvious symmetry are discussed. In contradistinction to the usual approach, our coupled system of equations leads to the simultaneous solutions for all four-point Green functions (elastic and inelastic) appearing in a given theory.

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Bethe-Salpeter equation [1] has been widely applied in the studies of two body scattering and bound state problems (an exhaustive review can be found in [2]; for more recent applications see [3]). Moreover it has been demonstrated ([4], [5] and references therein) that exact Bethe-Salpeter formalism in the complex momentum space - together with its analytical dependence on both complex energy and complex angular momentum - follows from basic principles of massive QFT such as locality, Lorentz covariance and spectral condition. A possibility of extension of this rigorous approach to theories of QCD-type with discrete spectrum of composite particles ("quarkonia", "glueballs") appearing as Regge-type particles has been recently announced [5]. Having in mind these impressive efforts and developments it seems rather surprising that the general form of Bethe-Salpeter equation for two-body multichannel problems has not been discussed. In this context let us mention here that there are indications of both  $q\bar{q}$  and  $gg$  content in glueballs [6] and that two-body coupled channel 3-D formalism has been successfully applied recently in phenomenology of hadron interactions [7], [8]. It seems therefore appropriate to start discussion of multichannel Bethe-Salpeter equations in the literature. Our contribution to this subject will be a discussion of some general properties of the solutions followed by an example of the multichannel solution of the special problem i.e. finding positions and couplings of Regge poles in the vicinity of  $\ell = -1$  in a theory with the most general trilinear couplings of  $N$  different scalar fields. A generalization of Bethe-Salpeter equation for the two body multichannel off-shell amplitude is formally straightforward:

$$M = B + B \bullet G \bullet M \quad (1)$$

$B, G, M$  being matrices in both continuous and discrete indices i.e.  $M \equiv M(a, k_1; b, k_2 | a', k'_1; b', k'_2)$  with  $k_1 + k_2 = k'_1 + k'_2$  denotes off-shell extrapolations of the scattering amplitude for the process with incoming particles  $a$  and  $b$  with momenta  $k_1$  and  $k_2$  respectively and outgoing  $a'(k'_1)$  and  $b'(k'_2)$ .  $B \equiv B(a, k_1; b, k_2 | a', k'_1; b', k'_2)$  is a two-particle irreducible kernel and  $G$ , up to a constant factor, is multiplication of two, two-point Green functions:

$$G(a, k_1; b, k_2 | a', k'_1; b', k'_2) \sim G_F(a, k_1; a', k'_1) G_F(b, k_2; b', k'_2) \delta^4(k_1 - k'_1) \delta^4(k_2 - k'_2) \quad (2)$$

Symbol  $\bullet$  denotes matrix multiplication both in discrete and continuous indices. Two-point functions  $G_F$  need not be diagonal in discrete indices. In such a manner we admit both spinor propagators and eventual mixing of different elementary fields. If we had complete knowledge of matrix functions  $B, G$  in the euclidean region done, and were able to solve eqn.(1) there, then the basic principles of QFT would guarantee [4], [5] that this solution - analytically continued as a known function of invariants - yields all available in a given theory information on the scattering matrix function  $M$  in the physical region. Generally none of these conditions is met and one has to rely on soundness of chosen scheme of approximations. Having this in mind we shall write below a few general relations which in our opinion may be helpful at least as a consistency check of approximations. To begin with, let us remind, that for a given  $B, G$  solution of eqn.(1)  $M\{B, G\}$  is given as:

$$M\{B, G\} = [1 - B \bullet G]^{-1} \bullet B \quad (3)$$

or equivalently:

$$M\{B, G\} = B \bullet [1 - G \bullet B]^{-1} \quad (4)$$

This solution, treated as a functional of  $B$  and  $G$  has the following interesting property:  
For arbitrary  $G_1$  one has

$$M\{B, G\} = M\{M\{B, G - G_1\}, G_1\} \quad (5)$$

This equality has been found useful for 1-channel problem long time ago [9]. In fact equation (5) comes as an answer to the following question: what is the relation between pairs  $(B, G)$  and  $(B_1, G_1)$  if both of them lead to the same  $M$ :

$$M\{B, G\} = M\{B_1, G_1\} \quad (6)$$

Using eqns.(3), (4), (6) we get  $B_1 = B + B \bullet [G - G_1] \bullet B_1$  i.e. from (1) we have  $B_1 = M\{B, G - G_1\}$ ; inserting it into (6) we end up with relation (5).

Eqn.(5) leads to the following formula for  $G = G_1 + G_2 + G_3 + \dots + G_n$ ,

$$M\{B, G\} = M\{\dots M\{M\{M\{B, G_1\}, G_2\}, G_3\} \dots, G_n\} \quad (7)$$

It is natural to ask whether similar formula holds in the situation, when we subdivide  $B$  instead of  $G$ . The answer is yes, once we notice that the following relations between off-shell scattering amplitude  $M\{B, G\}$  and non-amputated 4-point Green function  $M\{G, B\}$  hold:

$$M\{B, G\} = B + B \bullet M\{G, B\} \bullet B \quad (8)$$

$$M\{G, B\} = G + G \bullet M\{B, G\} \bullet G \quad (9)$$

Inserting (7) with  $G \leftrightarrow B$  in r.h.s. of (8) we get for  $B = \sum_{k=1}^n B_k$ ,

$$M\{B, G\} = B + B \bullet M\{\dots M\{M\{M\{G, B_1\}, B_2\}, B_3\} \dots, B_n\} \bullet B \quad (10)$$

The formulae (7 - 10) may provide useful schemes of approximation for subdivision of Green functions and irreducible Bethe-Salpeter kernels. Of course in principle these two schemes can be combined. Let us mention that approaches with suitable divisions of  $G$  and/or  $B$  have already been useful both in the case of QED applications [10] and for deriving rigorous results concerning threshold behaviour in QFT [11]. The essential relation found in [11] exhibits convenience of using half-amputated amplitudes and kernels  $B_H, M_H$ :

$$M_H = M\{B, G\} \bullet G, \quad B_H = B \bullet G \quad (11)$$

For this case formula (1) yields  $M_H = M\{B_H, 1\}$ . Moreover, from eqn.(8)

$$M\{1, B_H\} = 1 + M\{B_H, 1\} \quad (12)$$

Using division of  $B_H$  in  $M\{1, B_H\}$  together with eqns.(5),(12) we recover in our notation Bros-Iagolnitzer relation (compare text before eqn.(5) of ref.[11]):

$$(1 + M \bullet G) = (1 + M'' \bullet G) \bullet (1 - B' \bullet G \bullet (1 + M'' \bullet G))^{-1} \quad (13)$$

where  $M = M\{B, G\}$ ,  $M'' = M\{B - B', G\}$ .

In this note we shall not investigate in detail how useful our eqns.(7), (10) are in practice.

We shall pass instead to a problem where the use of multichannel approach itself can be clearly exhibited and discussed in QFT. We illustrate this on an example of theory with trilinear couplings. The most general trilinear coupling of  $N$  hermitian scalar fields,  $\varphi_i (i = 1, \dots, N)$  is of the form:

$$\mathcal{L}_{int}(x) = \frac{1}{3!} \sum_{i,j,k=1}^N c_{ijk} \varphi_i(x) \varphi_j(x) \varphi_k(x) \quad (14)$$

The totally symmetric symbol  $c_{ijk}$  denotes the coupling constants.

Let us start with the off-shell amplitude:

$$T(q_1, i_1; q_3, i_3 | q_2, i_2; q_4, i_4) = (2\pi)^4 \delta(q_1 + q_3 - q_2 - q_4) M_{i_3 i_4}^{i_1 i_2}(P_{tot}, q_{13}, q_{24}) \quad (15)$$

with  $P_{tot} = q_1 + q_3 = q_2 + q_4$  and  $q_{ij} = \frac{q_i - q_j}{2}$ .

The Bethe-Salpeter equation for  $M_{i_3 i_4}^{i_1 i_2}(P_{tot}, q_{13}, q_{24})$  in the ladder approximation reads:

$$M_{i_3 i_4}^{i_1 i_2}(P_{tot}, q_{13}, q_{24}) = B_{i_3 i_4}^{i_1 i_2}(q_{13}, q_{24}) + \sum_{j,k} \int d^4 q B_{i_3 k}^{i_1 j}(q_{13}, q) \mathcal{G}_k^j(P_{tot}, q) M_{k i_4}^{j i_2}(P_{tot}, q, q_{24}) \quad (16)$$

where  $B_{i_3 k}^{i_1 j}(q_{13}, q) = \sum_i c_{i_1 i j} c_{i_3 i k} / [(q_{13} - q)^2 - m_i^2 + i\varepsilon]$  and  $\mathcal{G}_k^j(P_{tot}, q)$  corresponds to diagonal terms of  $G$  from eq.(2)

$$\mathcal{G}_k^j(P_{tot}, q) = \frac{i}{(2\pi)^4} \frac{1}{\left(\frac{P_{tot}}{2} + q\right)^2 - m_j^2 + i\varepsilon} \frac{1}{\left(\frac{P_{tot}}{2} - q\right)^2 - m_k^2 + i\varepsilon} \quad (17)$$

In the c. m. frame ( $P_{tot} = (W, 0)$ ) the partial-wave decomposition of  $M_{i_3 i_4}^{i_1 i_2}$  is given by

$$M_{i_3 i_4}^{i_1 i_2}(P_{tot}^{c.m.}, q_{13}, q_{24}) = \frac{1}{4\pi |\vec{q}_{13}| |\vec{q}_{24}|} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(\cos \vartheta) M_{\ell}^{i_1, i_3; i_2, i_4}(W, q_{13}^0, |\vec{q}_{13}|, q_{24}^0, |\vec{q}_{24}|) \quad (18)$$

where  $\vartheta$  is the angle between  $\vec{q}_{13}$  and  $\vec{q}_{24}$ .

In what follows we shall simplify the notation, introducing an index  $I(r, s)$  characterizing the channel  $(r, s)$ , i. e.

$$(r, s) \rightarrow I(r, s) = 1, \dots, N^2 \quad , \quad (19)$$

and define matrices  $\mathbf{M}_{\ell IJ}$ ,  $\mathbf{B}_{\ell IJ}$  and  $\mathbf{G}_{IJ}$  by

$$\mathbf{M}_{\ell I(i,j)I(k,l)}(W, q_{13}^0, |\vec{q}_{13}|, q_{24}^0, |\vec{q}_{24}|) = M_{\ell}^{i,j;k,l}(W, q_{13}^0, |\vec{q}_{13}|, q_{24}^0, |\vec{q}_{24}|) \quad , \quad (20)$$

$$\mathbf{B}_{\ell I(i,j)I(k,l)}(q_{13}^0, |\vec{q}_{13}|, q_{24}^0, |\vec{q}_{24}|) = -2\pi \sum_{r=1}^N c_{irk} c_{jrl} Q_{\ell}(\beta_r(q_{13}^0, |\vec{q}_{13}|, q_{24}^0, |\vec{q}_{24}|)) \quad , \quad (21)$$

$$\mathbf{G}_{I(i,j)I(k,l)}(W, k^0, |\vec{k}|, q^0, |\vec{q}|) = \delta_{I(i,j)I(k,l)} \delta(k^0 - q^0) \delta(|\vec{k}| - |\vec{q}|) \mathcal{G}_j^i(P_{tot}, q) \quad . \quad (22)$$

In (21) the variable  $\beta_r$  is

$$\beta_r(q_{13}^0, |\vec{q}_{13}|, q_{24}^0, |\vec{q}_{24}|) = \frac{-(q_{13}^0 - q_{24}^0)^2 + \vec{q}_{13}^2 + \vec{q}_{24}^2 + m_r^2 - i\varepsilon}{2|\vec{q}_{13}| |\vec{q}_{24}|} \quad . \quad (23)$$

If the  $\bullet$  product for two such matrices is defined explicitly by

$$(X \bullet Y)_{IJ}(k^0, |\vec{k}|, q^0, |\vec{q}|) := \sum_{K=1}^{N^2} \int dq'^0 d|\vec{q}'| X_{IK}(k^0, |\vec{k}|, q'^0, |\vec{q}'|) Y_{KJ}(q'^0, |\vec{q}'|, q^0, |\vec{q}|) \quad , \quad (24)$$

then the Bethe-Salpeter equation for the partial wave amplitudes is of the form (1),

$$M_\ell = B_\ell + B_\ell \bullet G \bullet M_\ell \quad , \quad (25)$$

with the solution (4):

$$M_\ell = B_\ell \bullet \frac{1}{1 - G \bullet B_\ell} = M\{B_\ell, G\} \quad . \quad (26)$$

Here, one may split  $B_\ell$  according to (21) into terms corresponding to the exchange of particles of type  $r$ ,

$$B_\ell = \sum_{r=1}^N B_\ell^{(r)} \quad , \quad (27)$$

$B_{\ell I(i,j)I(k,l)}^{(r)} = -2\pi c_{irk} c_{jrl} Q_\ell(\beta_r)$ . In this way we are naturally led to a decomposition of the form (10). Thus,  $M_\ell$  can be written as

$$M_\ell = B_\ell + B_\ell \bullet M\{\dots M\{M\{G, B_\ell^{(1)}\}, B_\ell^{(2)}\}, \dots, B_\ell^{(N)}\} \bullet B_\ell \quad . \quad (28)$$

This form of the solution may be useful for numerical studies of  $M_\ell$  because the contributions of the various exchanged particles are now disentangled in a systematic way.

We will in the following focus our interest on Regge trajectories near  $\ell = -1$ ; such a choice seems sound once we recall that studies of this singularity in  $\varphi^3$  theory (i. e. only  $c_{111} \neq 0$  in eq. (14)) often were and still are a point of departure for studies of the diffractive region in QCD (see, e. g. [12]). In our discussion we shall not go beyond the approximation of leading logarithms which means that—in analogy with the one-channel case [13]— $Q_\ell$  can be replaced by its leading  $\ell$ -plane singularity,  $Q_\ell(\beta) \rightarrow 1/(\ell + 1)$ . In this case, the matrix  $B_\ell$  becomes constant,

$$B_{\ell I(i,j)I(k,l)}(k^0, |\vec{k}|, q^0, |\vec{q}|) \rightarrow -\frac{2\pi}{\ell + 1} C_{I(i,j)I(k,l)} = -\frac{2\pi}{\ell + 1} \sum_{r=1}^N c_{irk} c_{jrl} = \text{const} \quad , \quad (29)$$

so that

$$M_\ell = -2\pi C \bullet \frac{1}{\ell + 1 + 2\pi G \bullet C} \quad . \quad (30)$$

Now, since  $C$  does not depend on the variables that are integrated by the  $\bullet$  product, (30) simplifies to give a usual matrix equation: One finds

$$C \bullet G \bullet C = -\frac{1}{2\pi} CFC \quad (31)$$

and so on, with the diagonal matrix  $F$  given by

$$F_{I(i,j)I(k,l)}(W) = -2\pi \delta_{I(i,j)I(k,l)} \int dq^0 d|\vec{q}| \mathcal{G}_j^i(P_{tot}, q) \equiv \delta_{I(i,j)I(k,l)} F_{I(i,j)}(W) \quad . \quad (32)$$

Therefore, the solution of the Bethe-Salpeter equation is

$$\mathbf{M}_\ell = -2\pi\mathbf{C}\frac{1}{\ell+1-\mathbf{F}\mathbf{C}} \quad . \quad (33)$$

Here everything is to be understood as a matrix relation. We will in the following study the high-energy behavior of the amplitude in the crossed channel. To do this, we define the symmetric matrix  $\hat{\mathbf{C}} = \sqrt{\mathbf{F}}\mathbf{C}\sqrt{\mathbf{F}}$  and write (33) in the form

$$\mathbf{M}_\ell = -2\pi\sqrt{\mathbf{F}}^{-1}\hat{\mathbf{C}}\frac{1}{\ell+1-\hat{\mathbf{C}}}\sqrt{\mathbf{F}}^{-1} \quad . \quad (34)$$

We shall limit our considerations to  $P_{tot}^2 \leq 0$  so that  $F_{IJ}$  comes out to be real and positive.

Since  $\hat{\mathbf{C}}$  is symmetric, we can diagonalize it: Let  $\mathbf{v}^{[i]}$  denote the normalized eigenvectors,

$$\hat{\mathbf{C}}\mathbf{v}^{[i]} = \lambda^{[i]}\mathbf{v}^{[i]} \quad , \quad i = 1, \dots, N^2 \quad , \quad (35)$$

then

$$\mathbf{M}_\ell = -2\pi\sqrt{\mathbf{F}}^{-1} \left( \sum_i \frac{\mathbf{v}^{[i]}\lambda^{[i]}\mathbf{v}^{[i]\dagger}}{\ell+1-\lambda^{[i]}} \right) \sqrt{\mathbf{F}}^{-1} \quad . \quad (36)$$

Thus, the Regge trajectories are given by

$$\alpha^{[i]} = \ell_{\text{pole}} = \lambda^{[i]} - 1 \quad . \quad (37)$$

With (36) we are ready to use the Mandelstam-Sommerfeld-Watson representation [13, 14, 15] to obtain the high-energy amplitude in the crossed channel ( $t = (q_1 + q_3)^2 = W^2$  fixed,  $s = (q_1 - q_2)^2 \rightarrow \infty$ ), with the result

$$\mathbf{M}_{\text{poles}} \stackrel{s \rightarrow \infty, t \text{ fixed}}{\equiv} \frac{1}{s}\sqrt{\mathbf{F}}^{-1} \left( \sum_i \mathbf{v}^{[i]}\lambda^{[i]}s^{\lambda^{[i]}}\mathbf{v}^{[i]\dagger} \right) \sqrt{\mathbf{F}}^{-1} = \frac{1}{s}\sqrt{\mathbf{F}}^{-1} \left( \hat{\mathbf{C}} \exp(\hat{\mathbf{C}} \ln s) \right) \sqrt{\mathbf{F}}^{-1} \quad . \quad (38)$$

The states  $\mathbf{v}^{[i]}$  are eigenstates of the hermitian matrix  $\hat{\mathbf{C}}$ , thus they are orthogonal. This is, however, no longer true for the states  $\sqrt{\mathbf{F}}^{-1}\mathbf{v}^{[i]}$ . As the dual base for these states we introduce the states

$$\mathbf{u}^{[i]} = \sqrt{\mathbf{F}}\mathbf{v}^{[i]} \quad \text{with} \quad \mathbf{u}^{[i]\dagger}\mathbf{F}^{-1}\mathbf{u}^{[j]} = \delta_{ij} \quad . \quad (39)$$

The only matrix elements of  $\mathbf{M} = \mathbf{M}_{\text{poles}}$  in this base that are non-zero are  $\mathbf{u}^{[i]\dagger}\mathbf{M}\mathbf{u}^{[i]}$ ,  $i = 1, \dots, N^2$ . The high-energy behaviour of each of these matrix elements is governed by only one trajectory,

$$\mathbf{u}^{[i]\dagger}\mathbf{M}\mathbf{u}^{[j]} = \delta_{ij}\lambda^{[i]}s^{\lambda^{[i]}-1} \quad , \quad (40)$$

although the states  $\mathbf{u}^{[i]}$  are in general not orthogonal. Equation (40) leads to the following sum rules for the scattering amplitudes:

$$\sum_{I,J=1}^{N^2} \mathbf{v}_I^{[i]}\mathbf{v}_J^{[j]}\sqrt{F_I F_J}\mathbf{M}_{IJ} = \delta_{ij}\lambda^{[i]}s^{\lambda^{[i]}-1} \quad . \quad (41)$$

Note that some of these relations simplify whenever  $\lambda^{[i]} = 0$ . In this case,

$$\mathbf{M}\mathbf{u}^{[i]} = 0 \quad \text{for } \lambda^{[i]} = 0 \quad (42)$$

(or equivalently  $\mathbf{C}\mathbf{u}^{[i]} = 0$ ), i. e. states corresponding to fixed Regge singularities are transparent in our approximation.

As an illustration, let us consider the interaction

$$\mathcal{L}_I = \frac{g_1}{2!} : \varphi^2 \sigma : + \frac{g_2}{3!} : \sigma^3 : \quad , \quad (43)$$

which is a special case of (14) for  $N = 2$ ,  $\varphi_1 = \varphi$ ,  $\varphi_2 = \sigma$ , and  $c_{112} = c_{121} = c_{211} = g_1$ ,  $c_{222} = g_2$ ,  $c_{111} = c_{122} = c_{212} = c_{221} = 0$ . Let  $I(1, 1) = 1$ ,  $I(2, 2) = 2$ ,  $I(1, 2) = 3$ , and  $I(2, 1) = 4$ , then the matrix  $\mathbf{C}$  is block diagonal due to symmetry of total Lagrangian with respect to  $\varphi \rightarrow -\varphi$ ,

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}' & 0 \\ 0 & \mathbf{C}'' \end{pmatrix} \quad , \quad \mathbf{C}' = \begin{pmatrix} g_1^2 & g_1^2 \\ g_1^2 & g_2^2 \end{pmatrix} \quad , \quad \mathbf{C}'' = \begin{pmatrix} g_1 g_2 & g_1^2 \\ g_1^2 & g_1 g_2 \end{pmatrix} \quad , \quad (44)$$

and  $\mathbf{F}$  is diagonal with  $F_4 = F_3$ , so the matrix  $\hat{\mathbf{C}}$  is

$$\hat{\mathbf{C}} = \begin{pmatrix} \hat{\mathbf{C}}' & 0 \\ 0 & \hat{\mathbf{C}}'' \end{pmatrix} \quad , \quad \hat{\mathbf{C}}' = \begin{pmatrix} F_1 g_1^2 & \sqrt{F_1 F_2} g_1^2 \\ \sqrt{F_1 F_2} g_1^2 & F_2 g_2^2 \end{pmatrix} \quad , \quad \hat{\mathbf{C}}'' = F_3 \mathbf{C}'' \quad . \quad (45)$$

Diagonalizing  $\hat{\mathbf{C}}'$ ,  $\hat{\mathbf{C}}''$  we get eigenvalues

$$\lambda^{[1,2]} = \frac{1}{2} \left( F_1 g_1^2 + F_2 g_2^2 \pm \sqrt{(F_1 g_1^2 - F_2 g_2^2)^2 + 4 F_1 F_2 g_1^4} \right) \quad (46)$$

of  $\hat{\mathbf{C}}'$  and

$$\lambda^{[3,4]} = F_3 g_1 (g_2 \pm g_1) \quad , \quad (47)$$

of  $\hat{\mathbf{C}}''$ . The corresponding eigenvectors are

$$\mathbf{v}^{[1]} = \frac{1}{\sqrt{d}} \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \\ 0 \\ 0 \end{pmatrix} \quad , \quad \mathbf{v}^{[2]} = \frac{1}{\sqrt{d}} \begin{pmatrix} \sqrt{b} \\ -\sqrt{a} \\ 0 \\ 0 \end{pmatrix} \quad , \quad \mathbf{v}^{[3]} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \mathbf{v}^{[4]} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad ,$$

with  $d = \sqrt{(F_1 g_1^2 - F_2 g_2^2)^2 + 4 F_1 F_2 g_1^4}$ ,  $a = (d + F_1 g_1^2 - F_2 g_2^2)/2$ , and  $b = (d - F_1 g_1^2 + F_2 g_2^2)/2$ . Out of the ten relations described by (41), four are trivial due to the fact that  $\mathbf{M}$  is block diagonal. The remaining relations read (using the symmetry of  $\mathbf{M}$ )

$$a F_1 \mathbf{M}_{\varphi, \varphi \rightarrow \varphi, \varphi} + 2 F_1 F_2 g_1^2 \mathbf{M}_{\varphi, \sigma \rightarrow \varphi, \sigma} + b F_2 \mathbf{M}_{\sigma, \sigma \rightarrow \sigma, \sigma} = (\lambda^{[1]} - \lambda^{[2]}) \lambda^{[1]} s^{\lambda^{[1]}-1} \quad , \quad (48)$$

$$b F_1 \mathbf{M}_{\varphi, \varphi \rightarrow \varphi, \varphi} - 2 F_1 F_2 g_1^2 \mathbf{M}_{\varphi, \sigma \rightarrow \varphi, \sigma} + a F_2 \mathbf{M}_{\sigma, \sigma \rightarrow \sigma, \sigma} = (\lambda^{[1]} - \lambda^{[2]}) \lambda^{[2]} s^{\lambda^{[2]}-1} \quad , \quad (49)$$

$$\mathbf{M}_{\varphi, \varphi \rightarrow \sigma, \sigma} + 2 \mathbf{M}_{\varphi, \sigma \rightarrow \sigma, \varphi} + \mathbf{M}_{\sigma, \sigma \rightarrow \varphi, \varphi} = 2 g_1 (g_1 + g_2) s^{\lambda^{[3]}-1} \quad , \quad (50)$$

$$\mathbf{M}_{\varphi, \varphi \rightarrow \sigma, \sigma} - 2 \mathbf{M}_{\varphi, \sigma \rightarrow \sigma, \varphi} + \mathbf{M}_{\sigma, \sigma \rightarrow \varphi, \varphi} = 2 g_1 (g_2 - g_1) s^{\lambda^{[4]}-1} \quad , \quad (51)$$

$$F_1 g_1^2 \mathbf{M}_{\varphi, \varphi \rightarrow \varphi, \varphi} + (F_2 g_2^2 - F_1 g_1^2) \mathbf{M}_{\varphi, \sigma \rightarrow \varphi, \sigma} - F_2 g_1^2 \mathbf{M}_{\sigma, \sigma \rightarrow \sigma, \sigma} = 0 \quad , \quad (52)$$

$$M_{\varphi,\varphi\rightarrow\sigma,\sigma} - M_{\sigma,\sigma\rightarrow\varphi,\varphi} = 0 \quad , \quad (53)$$

where, e. g.,  $M_{\varphi,\sigma\rightarrow\varphi,\sigma}$  is short for  $M((\vec{p}_1, \varphi), (\vec{p}_2, \sigma) \rightarrow (\vec{p}_3, \varphi), (\vec{p}_4, \sigma))$ .

Of course, these relations are only valid in the Regge limit  $s \rightarrow \infty$ ,  $t$  fixed, and apply only to Regge behaviour, so that a vanishing combination of amplitudes means that this combination does not exhibit Regge behaviour.

In the example, eigenvalues  $\lambda^{[i]}$  can vanish only for  $g_2 = \pm g_1$ . If, say,  $g_2 = g_1$  (i. e.  $\lambda^{[1]} = g_1^2(F_1 + F_2)$ ,  $\lambda^{[2]} = 0$ ,  $\lambda^{[3]} = 2g_1^2 F_3$ ,  $\lambda^{[4]} = 0$ ), the relations (48) – (53) simplify to

$$\begin{aligned} M_{\varphi,\varphi\rightarrow\varphi,\varphi} &= M_{\sigma,\sigma\rightarrow\sigma,\sigma} = M_{\varphi,\sigma\rightarrow\varphi,\sigma} = g_1^2 s^{\lambda^{[1]}-1} \quad , \\ M_{\varphi,\varphi\rightarrow\sigma,\sigma} &= M_{\sigma,\sigma\rightarrow\varphi,\varphi} = M_{\varphi,\sigma\rightarrow\sigma,\varphi} = g_1^2 s^{\lambda^{[3]}-1} \quad . \end{aligned} \quad (54)$$

If moreover, we assumed  $m_\sigma = m_\varphi$ , then  $\lambda^{[1]} = \lambda^{[3]}$  and all the amplitudes standing on the l.h.s. of (54) are the same. This symmetry can be traced back to the fact, that for  $g_1 = g_2 \equiv g$  we can rewrite eqn.(43) as a sum of two separate  $\chi_\pm^3$  terms

$$\mathcal{L}_I(g_1 = g_2 \equiv g) = \frac{g}{2} \frac{1}{3!} \left[ : \chi_+^3 : + : \chi_-^3 : \right] \quad (55)$$

where  $\chi_\pm = (\sigma \pm \varphi)$ .

In the special case  $m_\sigma = m_\varphi$  total Lagrangian can be separated into two of identical form and hence the complete symmetry follows.

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